Recall the WZW action ($g: \Sigma \to G$):

$$I(g) = -\frac{i}{4\pi} \int_\Sigma d^2\sigma \sqrt{\rho}\,\rho^{ij}\, \mathrm{Tr}\left(g^{-1}\partial_i g \cdot g^{-1}\partial_j g\right) - i\Gamma(g)$$

where ρ is a metric on Σ and Γ is the Wess-Zumino term ($\partial B = \Sigma$):

$$\Gamma(g) = \int_B g^* \omega$$

$$\omega = \frac{1}{12\pi} \mathrm{Tr}\left(g^{-1}dg \wedge g^{-1}dg \wedge g^{-1}dg\right)$$

$$= \frac{1}{12\pi} \int_B d^3\sigma\, \varepsilon^{ijk}\, \mathrm{Tr}\, g^{-1}\partial_i g \cdot g^{-1}\partial_j g \cdot g^{-1}\partial_k g$$

The partition function of the WZW model is formally defined as a path integral:

$$Z = \int \mathcal{D}g\, e^{-\kappa I}$$

conformal invariance $\longrightarrow$ Z depends only on complex str. determined by ρ.

conformal block representation:

$$Z = \sum_i f_i \bar{f}_i = (f, f)$$

where f_i satisfy the KZ equation.

global symmetry:

The WZW action is invariant under the action of $G \times G = G_L \times G_R$:

$$g \longmapsto a g b^{-1}, \quad a \in G_L, \; b \in G_R$$

gauging WZW models:

take g to be section of a bundle $X \longrightarrow \Sigma$ with fiber G and structure group $G_L \times G_R$ or a subgroup (instead of a map $g: \Sigma \longrightarrow G$)

condition for gauging:

existence of "anomaly free" subgroup F of $G_L \times G_R$: $\mathrm{Tr}_L\, t t' - \mathrm{Tr}_R\, t t' = 0, \; t, t' \in \mathcal{F}$ $(*)$

($\mathcal{F}$ is Lie algebra of F).

Holomorphic wave-function:

take $F = G_R$ and define

$$I(g, A) = I(g) + \frac{1}{4\pi} \int_\Sigma d^2 z \, \mathrm{Tr}\, A_{\bar z}\, g^{-1} \partial_z g - \frac{1}{8\pi} \int_\Sigma d^2 z \, \mathrm{Tr}\, A_{\bar z} A_z$$

Then, under an infinitesimal gauge trf.:

$$\delta g = -gu\,, \quad \delta A_i = -D_i u = -\partial_i u - [A_i, u],$$

one has

$$\delta I(g,A) = \frac{1}{8\pi}\int_\Sigma d^2z\, \mathrm{Tr}\, u\left(\partial_z A_{\bar z} - \partial_{\bar z} A_z\right)$$

$$= \frac{i}{8\pi}\int_\Sigma \mathrm{Tr}\, u\, dA, \qquad \text{(anomaly)}$$

as (*) is not obeyed

(exercise)

We now formally define a functional of A by

$$\Psi(A) = \int \mathcal{D}g\, e^{-kI(g,A)}$$

$$= \int \mathcal{D}g\, \exp\left(-k\,I(g) - \frac{k}{4\pi}\int_\Sigma d^2z\, \mathrm{Tr}\, A_{\bar z}\, g^{-1}\partial_z g + \frac{k}{8\pi}\int_\Sigma d^2z\, \mathrm{Tr}\, A_{\bar z} A_z\right)$$

Ψ obeys two key equations:

1) $\left(\dfrac{\delta}{\delta A_z} - \dfrac{k}{8\pi} A_{\bar z}\right)\Psi = 0 \quad \leftarrow$ (exercise)

2) $\left(D_{\bar z}\dfrac{\delta}{\delta A_{\bar z}} + \dfrac{k}{8\pi} D_{\bar z} A_z - \dfrac{k}{4\pi} F_{\bar z z}\right)\Psi = 0$

Hint for proving 2):

By differentiating under the integral sign, the left-hand side of 2) equals:

$$-\frac{k}{4\pi}\int \mathcal{D}g\, e^{-kI(g,A)} \underbrace{\left[D_{\bar z}\left(g^{-1}D_z g\right)+F_{\bar z z}\right]}_{=\,\delta_g I(g,A)}$$

where we have introduced the covariant derivative $Dg = dg - gA$.

$\longrightarrow$ path integral vanishes by integration by parts in g space.

Using the covariant derivatives introduced in the last lecture,

$$\frac{D}{DA_z} = \frac{\delta}{\delta A_z} - \frac{k}{8\pi}A_{\bar z}\,, \quad \frac{D}{DA_{\bar z}} = \frac{\delta}{\delta A_{\bar z}} + \frac{k}{8\pi}A_z\,,$$

we see that Ψ satisfies

$$1) \rightarrow \frac{D}{DA_z}\Psi = 0$$

$$2) \longrightarrow \left(D_{\bar z}\frac{D}{DA_{\bar z}} - \frac{k}{4\pi}F_{\bar z z}\right)\Psi = 0$$

$\longrightarrow$ Ψ is Chern-Simons wave-function!

The norm of the wave-function:

Define for every complex Riemann surface Σ, a vector space V consisting of hol. gauge inv. sections of $\mathcal{L}^{\otimes k}$ over $\mathcal{A}$.

A natural Hermitian structure on V is given by

$$(\Psi_1, \Psi_2) = \frac{1}{\mathrm{vol}(\hat{G})} \int_{\mathcal{A}} \mathcal{D}A\, \overline{\Psi_1}(A)\, \Psi_2(A)$$

To evaluate this pairing, let us first define $\overline{\Psi}$:

$$h: \Sigma \longrightarrow G$$

B gauge field gauging subgroup of G_L in $G_L \times G_R$

$$\longrightarrow I'(h, B) = I(h) - \frac{1}{4\pi} \int_\Sigma d^2z \,\mathrm{Tr}\, B_z \partial_{\bar{z}} h \cdot h^{-1}$$

$$- \frac{1}{8\pi} \int_\Sigma d^2z \,\mathrm{Tr}\, B_z B_{\bar{z}}$$

Under $\quad \delta h = uh, \quad \delta B_i = -D_i u,$

one has

$$\delta \underline{I}'(h, B) = -\frac{1}{8\pi}\int_\Sigma d^2z \, \mathrm{Tr}\, u\left(\partial_z B_{\bar z} - \partial_{\bar z} B_z\right)$$

Then define

$$\chi(B) = \int \mathcal{D}h \, e^{-k\underline{I}'(h,B)}$$

We have $\chi(A) = \overline{\Psi(A)}$

We compute

$$|\Psi|^2 = \frac{1}{\mathrm{vol}(\hat{G})}\int_{\mathcal{A}} DA \, \overline{\Psi(A)} \, \Psi(A)$$

$$= \frac{1}{\mathrm{vol}(\hat{G})}\int \mathcal{D}g \, \mathcal{D}h \, \mathcal{D}A \, \exp\Big(-k\underline{I}(g) - k\underline{I}(h)$$

$$-\frac{k}{4\pi}\int_\Sigma d^2z \, \mathrm{Tr}\, A_{\bar z} g^{-1}\partial_z g + \frac{k}{4\pi}\int_\Sigma d^2z \, \mathrm{Tr} A_z \partial_{\bar z} h \cdot h^{-1}$$

$$+ \frac{k}{4\pi}\int_\Sigma d^2z \, \mathrm{Tr}\, A_{\bar z} A_z \Big) \qquad (**)$$

Notice that the integrand is invariant under gauge transformations:

$$\delta g = -gu, \quad \delta h = uh, \quad \delta A_i = -D_i u$$

Since $(**)$ is quadratic in A, we can perform the Gaussian integral over A

to give

$$|\Psi|^2 = \frac{1}{\mathrm{vol}(\hat{G})} \int \mathcal{D}g\, \mathcal{D}h \exp\Big(-\kappa I(g) - \kappa I(h) + \frac{\kappa}{2\pi} \int_\Sigma d^2z \operatorname{Tr} g^{-1}\partial_z g\, \partial_{\bar z} h \cdot h^{-1}\Big)$$

(exercise)

Using the Polyakov-Wiegman formula,

$$I(gh) = I(g) + I(h) - \frac{1}{2\pi} \int_\Sigma d^2z \operatorname{Tr} g^{-1}\partial_z g\, \partial_{\bar z} h \cdot h^{-1},$$

one can replace the double integral over g and h by a single integral over $f = gh$

$\longrightarrow$ cancel $\mathrm{vol}(\hat{G})$

$$\longrightarrow \quad |\Psi|^2 = \int \mathcal{D}f\, e^{-\kappa I(f)} = Z(\Sigma)$$

partition function of WZW model

<u>Varying the complex structure of Σ</u>

$\Psi(A_i \rho)$ becomes a section of the bundle

$$\begin{array}{ccc} \tilde{\mathcal{H}}_{Q_0} & \longrightarrow & \mathcal{V} \\ & & \downarrow \\ & & \tilde{\mathcal{T}} \end{array}$$

$\Psi(A;\rho)$ is anti-holomorphic if it is annihilated by

$$\nabla^{(1,0)} = \delta^{(1,0)} + \frac{4\pi}{K}\int_\Sigma \delta\rho_{\bar z\bar z}\, \mathrm{Tr}\frac{D}{DA_{\bar z}}\frac{D}{DA_{\bar z}}$$

We compute

$$\delta^{(1,0)}\Psi = \int \mathcal{D}g\, e^{-\kappa I(A,g)}\left(-\frac{K}{4\pi}\int_\Sigma d^2z\, \delta\rho_{\bar z\bar z}\,\rho^{\bar z z}\,\mathrm{Tr}(g^{-1}D_z g)^2\right),$$

(exercise)

where $D_i g = \partial_i g - gA_i$. Similarly,

$$\frac{D}{DA_{\bar z}}\Psi = \int \mathcal{D}g\, e^{-\kappa I(g,A)}\cdot\frac{K}{4\pi}g^{-1}D_z g,$$

(exercise)

giving

$$\mathrm{Tr}\frac{D}{DA_{\bar z}}\frac{D}{DA_{\bar z}}\Psi = \int \mathcal{D}g\, e^{-\kappa I(g,A)}\cdot\left(\frac{K}{4\pi}\right)^2 \mathrm{Tr}(g^{-1}D_z g)^2$$

Combining, we thus obtain

$$\nabla^{(1,0)}\Psi = 0$$

Choosing an orthonormal basis of covariantly constant sections of $\mathcal{V}$, e_α, $\alpha = 1 \dots \dim \mathcal{H}_Q$,

$$\Psi(A;\rho) = \sum_\alpha e_\alpha(A;\rho)\cdot \overline{f_\alpha}(\rho) \Rightarrow Z(\Sigma;\rho) = \sum_{\alpha=1}^{\dim\mathcal{H}_Q} |f_\alpha|^2$$